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Research Article

Conjugacy Separability of Some One-Relator Groups

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Conjugacy separability of any group of the class of one-relator groups given by the presentation $\langle a, b; [a^m, b^n] = 1\rangle (m, n > 1)$ is proven. The proof made used of theoretical combinatorial group methods, namely the structure of amalgamated free products and some properties of the subgroups and quotients of any group of the class of one-relator groups given above.

1. Introduction

A group G is conjugacy separable if for any two nonconjugate elements f and g of G there exists a homomorphism φ of group G onto some finite group X such that the images $f\varphi$ and $g\varphi$ of elements f and g are not conjugate in group X. It is clear that any conjugacy separable group G is residually finite (i.e., recall that for any nonidentity element $g \in G$ there exists a homomorphism φ of group G onto some finite group X such that $g\varphi \neq 1$).

Conjugacy separability is related to the conjugacy problem, as the residual finiteness is related to the word problem in the study of groups. In fact, Mostowski [1] proved that finitely presented conjugacy separable groups have solvable conjugacy problem, just as finitely presented residually finite groups have solvable word problem.

Since 1962 when Baumslag and Solitar [2] discovered the first examples of nonresidually finite one-relator groups a lot of results establishing the residual finiteness of various one-relator groups have appeared. It was also shown that some of such groups are conjugacy separable. So, Dyer [2] has proved the conjugacy separability of any group that is free product of free groups amalgamating cycle. In the same paper she proved an unpublished result of M. Armstrong on conjugacy separability of any one-relator group with nontrivial center. The conjugacy separability of groups defined by relator of form $(a^m b^n)^t$ and $(a^{-1} b^l a b^m)^s$, where $s > 1$, was proved in [4] and [5], respectively. It should mention that up to now it is not known whether there exists one-relator group that is residually finite but not conjugacy separable.

In this paper we enlarge the class of conjugacy separable one-relator groups. Namely, we prove here the following theorem:

Theorem 1.1. *Any group* $G_{mn} = \langle a, b; \quad [a^m, b^n] = 1 \rangle$ *where integers* m *and* n *are greater than 1, is conjugacy separable.*

We note that the assertion of Theorem is also valid when $m = 1$ or $n = 1$. Indeed, in this case group G_{mn} is the generalized free product of two finitely generated abelian groups with cyclic amalgamation and its conjugacy separability follows from the result of [3].

The residual finiteness of groups G_{mn} is well known; it follows, for example, from the result of paper [6]. Groups G_{mn} are of some particular interest: they are centerless, torsion-free,... Some properties of these groups were considered in [7, 8] where, in particular, the description of their endomorphisms was given and their automorphisms group were studied respectively. In this paper, the proofs made use of presentation of group G_{mn} as amalgamated free product. This presentation will be the crucial tool in the proof of Theorem 1.1 and because of this, in Section 1, we recall the Solitar theorem on the conjugacy of elements of amalgamated free products and derive, for our special case, the criterion which is somewhat simpler. In Section 2 some properties of separability of groups G_{mn} are established and in Section 3 the proof of Theorem 1.1 will be completed.

We remind that conjugacy separability of groups G_{mn} can also be obtained using [9]. However, the method in [9] presents any group G_{mn} as the result of adjoining roots to conjugacy separable groups.

2. Preliminary Remarks on the Conjugacy in Amalgamated Free Products

Let us recall some notions and properties concerned with the construction of free product $G = (A * B; \quad H)$ of groups A and B with amalgamated subgroup H.

Every element $g \in G$ can be written in a form

$$g = x_1 x_2 \cdots x_r, \quad (r \geqslant 1) \tag{2.1}$$

where for any $i = 1, 2, \ldots, r$ element x_i belongs to one of the free factors A or B and if $r > 1$ any successive x_i and x_{i+1} do not belong to the same factor A or B (and therefore for any $i = 1, 2, \ldots, r$ element x_i does not belong to amalgamated subgroup H). Such form of element g is called *reduced*.

In general, element $g \in G$ can have different reduced forms, but if $g = y_1 y_2 \cdots y_s$ is one more reduced form of g, then $r = s$ and, for any $i = 1, 2, \ldots, r$, x_i and y_i belong to the same factor A or B. The *length* $l(g)$ of element g is then defined as the number r of the components in a reduced form of g. Element g is called *cyclically reduced* if either $l(g) = 1$ or $l(g) > 1$ and the first and the last components of its reduced form do not belong to the same factor A or B.

If $g = x_1x_2\cdots x_r$ is a reduced form of cyclically reduced element g then cyclic permutations of element g are elements of form $x_ix_{i+1}\cdots x_rx_1\cdots x_{i-1}$, where $i = 1, 2, \ldots, r$.

If X is a subgroup of a group Y we will say that elements a and b of Y are *X-conjugate* if $a = x^{-1}bx$, for some $x \in X$.

The Solitar criterion of conjugacy of elements of amalgamated free product of two groups (Theorem 4.6 in [10]) can be formulated as follows.

Proposition 2.1. *Let $G = (A * B; H)$ be the free product of groups A and B with amalgamated subgroup H. Every element of G is conjugate to a cyclically reduced element. If lengths of two cyclically reduced elements are nonequal then these elements are not conjugate in G. Let f and g be cyclically reduced elements of G such that $l(f) = l(g)$. Then*

(1) *if $f \in A$ and f is not conjugate in A to any element of subgroup H, then f and g are conjugate in group G if and only if $g \in A$ and f and g are conjugate in A; similarly, if $f \in B$ and f is not conjugate in B to any element of subgroup H then f and g are conjugate in group G if and only if $g \in B$ and f and g are conjugate in B;*

(2) *if $f \in H$ then f and g are conjugate in group G if and only if there exists a sequence of elements*

$$f = h_0, h_1, \ldots, h_n, h_{n+1} = g \tag{2.2}$$

such that for any $i = 0, 1, \ldots, n$ $h_i \in H$ and elements h_i and h_{i+1} are A-conjugate or B-conjugate;

(3) *If $l(f) = l(g) > 1$, then f and g are conjugate in group G if and only if element f is H-conjugate to some cyclic permutation of g.*

In the following special case assertion (2) of Proposition 2.1 became unnecessary.

Proposition 2.2. *Let $G = (A * B; H)$ be the free product of groups A and B with amalgamated subgroup H. Suppose that for any element f of A or B and for any element h of subgroup H, the inclusion $f^{-1}hf \in H$ is valid if and only if elements f and h commute. If f and g are cyclically reduced elements of G such that $l(f) = l(g) = 1$ then f and g are conjugate in group G if and only if f and g belong to the same subgroup A or B and are conjugate in this subgroup.*

Indeed, let elements f and g be conjugate in group G and let f belong to subgroup A, say. If f is not conjugate in A to any element of subgroup H, then the desired assertion is implied by assertion (1) of Proposition 2.1. Otherwise, we can assume that $f \in H$ and by assertion (2) of Proposition 2.1 there is a sequence of elements

$$f = h_0, h_1, \ldots, h_n, h_{n+1} = g, \tag{2.3}$$

such that for any $i = 0, 1, \ldots, n$ $h_i \in H$, and for suitable element x_i belonging to one of the subgroups A or B, the equality $x_i^{-1}h_ix_i = h_{i+1}$ holds. Since for $i = 0, 1, \ldots, n-1$ the inclusion $x_{i+1}^{-1}h_{i+1}x_{i+1} \in H$ is valid, the hypothesis gives $h_0 = h_n$, that is, $x_n^{-1}fx_n = g$. This means that elements f and g belong to that subgroup A or B which contains element x_n and also are conjugated in this subgroup.

Next, we describe more explicitly the situation arising in assertion (3) of Proposition 2.1.

Proposition 2.3. *Let $f = x_1x_2\cdots x_r$ and $g = y_1y_2\cdots y_r$ be the reduced forms of elements f and g of group $G = (A * B; H)$ where $r > 1$. Then f and g are H-conjugate if and only if there exist elements $h_0, h_1, h_2, \dots, h_r = h_0$ in subgroup H such that for any $i = 1, 2, \dots, r$, one has*

$$x_i = h_{i-1}^{-1} y_i h_i. \tag{2.4}$$

Proof. If for some elements $h_0, h_1, h_2, \dots, h_r = h_0$ of subgroup H equalities (2.4) hold, then

$$f = x_1x_2\cdots x_r = h_0^{-1}y_1h_1\ h_1^{-1}y_2h_2\cdots\ h_{r-1}^{-1}y_2h_r = h_0^{-1}(y_1y_2\cdots y_r)h_r = h_0^{-1}gh_0. \tag{2.5}$$

Conversely, by induction on r we prove that if for some $h \in H$ the equality $f = h^{-1}gh$ holds then there exists a sequence of elements $h_0, h_1, h_2, \dots, h_r = h_0$ of subgroup H satisfying the equalities (2.4) and such that $h_0 = h$.

Rewriting the equality $f = h^{-1}gh$ in the form

$$x_r^{-1}\cdots x_2^{-1}x_1^{-1}h^{-1}y_1y_2\cdots y_rh = 1, \tag{2.6}$$

we see that since the expression in the left part of it cannot be reduced, elements x_1 and y_1 must be contained in the same subgroup A or B and element $x_1^{-1}h^{-1}y_1$ must belong to subgroup H. Denoting this element by h_1^{-1}, we have $x_1 = h^{-1}y_1h_1$. If $r = 2$ then the equality above takes the form $x_2^{-1}h_1^{-1}y_2h = 1$, whence $x_2 = h_1^{-1}y_2h$. Therefore, setting $h_0 = h_2 = h$, we obtain in that case the desired sequence.

If $r > 2$ let us rewrite the equality $f = h^{-1}gh$ in the form

$$h^{-1}y_1h_1x_2\cdots x_r = h^{-1}y_1y_2\cdots y_rh. \tag{2.7}$$

This implies that if we set $f' = h^{-1}h_1x_2\cdots x_r$ and $g' = y_2\cdots y_r$, then $f' = h^{-1}g'h$. Since the length of elements f' and g' is equal to $r-1$, then by induction, there exists a sequence $h_1' = h$, $h_2', \dots, h_r' = h_1'$ of elements of subgroup H, such that $h^{-1}h_1x_2 = (h_1')^{-1}y_2h_2'$ and for any $i = 2, 3, \dots, r$

$$x_i = \left(h_{i-1}'\right)^{-1}y_ih_i'. \tag{2.8}$$

Since $x_2 = h_1^{-1}y_2h_2'$, then setting $h_i = h_i'$ for $i = 2, 3, \dots, r$, we obtain the desired sequence, and induction is completed. Proposition 2.3 is proven. □

We conclude this section with one more property of amalgamated free product.

Proposition 2.4. *Let $G = (A * B; H)$ be the free product of groups A and B amalgamating subgroup H where H lies in the center of each of groups A and B. Then for any element $f \in G$ not belonging to subgroup A one has $f^{-1}Af \cap A = H$.*

Proof. In fact, since subgroup H coincides with the center of group G (see [10, page 211]), the inclusion $H \subseteq f^{-1}Af \cap A$ is obvious. Conversely, let element $f^{-1}af$ belong to subgroup A where $a \in A$. Since $f \notin A$ then either $f \in B \setminus H$ or $l(f) > 1$ and (without loss of generality) the first component of reduced form of f belongs to B. In any case the assumption $a \notin H$ would imply that $l(f^{-1}af) > 1$. Thus, $a \in H$ and therefore $f^{-1}af = a \in H$. □

3. Some Properties of Groups G_{mn} and of Their Certain Quotients

In what follows, our discussion will make use of the presentation of group G_{mn} as an amalgamated free product of two groups. To describe such presentation, let H be the subgroup of group G_{mn}, generated by elements $c = a^m$ and $d = b^n$. Also, let A denote the subgroup of group G_{mn} generated by element a and subgroup H, and let B denote the subgroup of group G_{mn}, generated by element b and subgroup H. Then it can be immediately verified that H is the free abelian group with base c, d, group A is the free product $(\langle a \rangle * H; \langle a^m = c \rangle)$ of infinite cycle $\langle a \rangle$ and group H with amalgamation $\langle a^m \rangle$, group B is the free product $(\langle b \rangle * H; \langle b^n = d \rangle)$ of infinite cycle $\langle b \rangle$ and group H with amalgamation $\langle b^n \rangle$, and group G_{mn} is the free product $(A * B; H)$ of groups A and B with amalgamation H.

The same decomposition is satisfiable for certain quotients of groups G_{mn}. Namely, for any integer $t > 1$ let $G_{mn}(t)$ be the group with presentation

$$\left\langle a, b; [a^m, b^n] = 1,\ a^{mt} = b^{nt} = 1 \right\rangle \tag{3.1}$$

and ρ_t the natural homomorphism of groups G_{mn} onto $G_{mn}(t)$. Then it is easy to verify that subgroup $H(t) = H\rho_t$ of group $G_{mn}(t)$ is isomorphic to the quotient H/H^t (where, as usually, H^t consists of all elements h^t, $h \in H$), subgroup $A(t) = A\rho_t$ is the amalgamated free product $(\langle a; a^{mt} = 1 \rangle * H(t); a^m = cH^t)$ of cycle $\langle a; a^{mt} = 1 \rangle$ of order mt and group $H(t)$, subgroup $B(t) = B\rho_t$ is the amalgamated free product $(\langle b; b^{nt} = 1 \rangle * H(t); b^n = dH^t)$ of cycle $\langle b; b^{nt} = 1 \rangle$ of order nt and group $H(t)$, and group $G_{mn}(t)$ is the amalgamated free product $(A(t) * B(t); H(t))$.

These decompositions of groups G_{mn} and $G_{mn}(t)$ are assumed everywhere below, and such notions as free factor, reduced form, length of element and so on will refer to them.

Let us remark, at once, that since each of groups $A(t)$ and $B(t)$ is the amalgamated free product of two finite groups and group $G_{mn}(t)$ is the free product of groups $A(t)$ and $B(t)$ with finite amalgamation, it follows from results of [3] that for every t, group $G_{mn}(t)$ is conjugacy separable. So, to prove the conjugacy separability of group G_{mn} it is enough, for any nonconjugate elements f and g of G_{mn}, to find an integer t such that elements $f\rho_t$ and $g\rho_t$ are nonconjugate in group $G_{mn}(t)$.

Since in the decompositions of groups A and B as well, as of groups $A(t)$ and $B(t)$, into amalgamated free product stated above the amalgamated subgroups are contained in the centre of each free factor, by Proposition 2.4 we have the following proposition

Proposition 3.1. *For any element g of group A (or B) not belonging to subgroup H the equality $g^{-1}Hg \cap H = \langle c \rangle$ (resp., $g^{-1}Hg \cap H = \langle d \rangle$) holds. In particular, for any element g of group A or B and for any element h of subgroup H element $g^{-1}hg$ belongs to subgroup H if and only if elements g and h commute.*

Similarly, for any element g of group $A(t)$ (or $B(t)$) not belonging to subgroup $H(t)$ the equality $g^{-1}H(t)g \cap H(t) = \langle cH^t \rangle$ (resp., $g^{-1}H(t)g \cap H(t) = \langle dH^t \rangle$) holds. In particular, for any elements g of group $A(t)$ or $B(t)$ and h of subgroup $H(t)$, element $g^{-1}hg$ belongs to subgroup $H(t)$ if and only if elements g and h commute.

Propositions 2.1, 2.2 and 3.1 lead to the following criterion for conjugacy of elements of groups G_{mn} and $G_{mn}(t)$:

Proposition 3.2. *Let G be any of groups G_{mn} and $G_{mn}(t)$. Every element of G is conjugate to a cyclically reduced element. If lengths of two cyclically reduced elements are not equal then these elements are not conjugate in G. Let f and g be cyclically reduced elements of G such that $l(f) = l(g)$. Then*

(1) *If $l(f) = l(g) = 1$ then f and g are conjugate in group G if and only if f and g belong to the same free factor and are conjugate in this factor.*

(2) *If $l(f) = l(g) > 1$ then f and g are conjugate in group G if and only if element f is H-conjugate or $H(t)$-conjugate to some cyclic permutation of g.*

We now consider some further properties of groups G_{mn} and $G_{mn}(t)$. The following assertion is easily checked.

Proposition 3.3. *For any homomorphism φ of group G_{mn} onto a finite group X there exist an integer $t > 1$ and a homomorphism ψ of group $G_{mn}(t)$ onto group X such that $\varphi = \rho_t\psi$.*

Also, for any integers $t > 1$ and $s > 1$ such that t divides s there exists homomorphism $\varphi : G_{mn}(s) \to G_{mn}(t)$ such that $\rho_t = \rho_s\varphi$.

The same assertions are valid for groups A and B.

Proposition 3.4. *For any element g of group A or B, if g does not belong to subgroup H then there exists an integer $t_0 > 1$ such that for any positive integer t, divisible by t_0, element $g\rho_t$ does not belong to subgroup $H\rho_t$.*

Proof. We will assume that $g \in A$; the case when $g \in B$ can be treated similarly.

So, let element $g \in A$ do not belong to subgroup H and let $g = x_1x_2\cdots x_r$ be a reduced form of g (in the decomposition of group A into amalgamated free product).

If $r = 1$ then element g belongs to subgroup $\langle a\rangle$ that is, $g = a^k$ for some integer k. Since $g \notin H$, integer k is not divisible by m. Then for any integer $t > 0$, in group $A(t)$, element $g\rho_t$ of subgroup $\langle a; a^{mt} = 1\rangle$ cannot belong to the amalgamated subgroup (generated by element a^m) of the decomposition of group $A(t)$ and consequently cannot belong to the free factor $H(t) = H\rho_t$.

Let $r > 1$. Every component x_i of the reduced form of element g is either of form a^k, where integer k is not divisible by m, or of form c^kd^l where $l \neq 0$. If integer t_0 is chosen such that the exponent l of any component x_i of the second kind is not divisible by t_0, then for any t divisible by t_0 the image $x_i\rho_t = x_iH^t$ of such component will not belong to the amalgamated subgroup of group $A(t)$ (generated, let's remind, by element cH^t). Moreover, as above in case $r = 1$, the images of components of the first kind will not belong to the amalgamated subgroup. Therefore, the form

$$g\rho_t = (x_1\rho_t)(x_2\rho_t)\cdots(x_r\rho_t) \tag{3.2}$$

of element $g\rho_t$ is reduced in group $A(t)$ and since $r > 1$, $g\rho_t$ does not belong to the free factor $H(t) = H\rho_t$. The proposition is proven. □

Proposition 3.4 obviously implies the the following proposition.

Proposition 3.5. *For any element g of group G_{mn} there exists an integer $t_0 > 1$ such that for all positive integer t, divisible by t_0, the length of element $g\rho_t$ in group $G_{mn}(t)$ coincides with the length of element g in group G_{mn}.*

Proposition 3.6. *For any elements f and g of group A (or B) such that element f does not belong to the double coset HgH, there exists an integer $t_0 > 1$ such that for any positive integer t, divisible by t_0, element $f\rho_t$ does not belong to the double coset $H(t)(g\rho_t)H(t)$.*

Proof. We can again consider only the case when elements f and g belong to subgroup A. So, let us suppose that element $f \in A$ does not belong to the double coset HgH. In view of Proposition 3.4, it is enough to prove that there exists a homomorphism φ of A onto a finite group X such that element $f\varphi$ of X does not belong to the double coset $(H\varphi)(g\varphi)(H\varphi)$.

To this end let's consider the quotient group $\overline{A} = A/C$ of group A by its (central) subgroup $C = \langle c\rangle$. The image xC of an element $x \in A$ in group $\overline{A}$ will be denoted by $\overline{x}$.

It is obvious that group $\overline{A}$ is the (ordinary) free product of the cyclic group X of order m, generated by element $\overline{a}$, and the infinite cycle Y, generated by $\overline{d}$. The canonical homomorphism of group A onto group $\overline{A}$ maps subgroup H onto subgroup Y and consequently, the image of the double coset HgH is the double coset $Y\overline{g}Y$. Since $C \leqslant H$, the element $\overline{f}$ does not belong to this coset.

We can assume, without loss of generality, that any element $\overline{f}$ and $\overline{g}$, if it is different from identity, has reduced form the first and the last syllables of which do not belong to subgroup Y. Every Y-syllable of these reduced forms is of form $\overline{d}^k$ for some integer $k \neq 0$. Since the set M of all such exponents k is finite, we can choose an integer $t > 0$ such that for any $k \in M$ the inequality $t > 2|k|$ holds. Let's denote by $\widetilde{A}$ the factor group of group $\overline{A}$ by the normal closure of element $\overline{d}^t$. Group $\widetilde{A}$ is the free product of groups X and Y/Y^t and since different integers from M are not relatively congruent and are not congruent to zero modulo t, then the reduced forms of the images $\widetilde{f}$ and $\widetilde{g}$ of elements $\overline{f}$ and $\overline{g}$ in group $\widetilde{A}$ are the same as in group $\overline{A}$. In particular, element $\widetilde{f}$ does not belong to the double coset $Y/Y^t\ \widetilde{g}\ Y/Y^t$. Since this coset consists of a finite number of elements and group $\widetilde{A}$ is residually finite, then there exists a normal subgroup $\widetilde{N}$ of finite index of group $\widetilde{A}$ such, that

$$\widetilde{f} \notin \left(Y/Y^t \widetilde{g} Y/Y^t\right) \cdot \widetilde{N}. \tag{3.3}$$

If now θ is the product of the canonical homomorphisms of group A onto group $\overline{A}$ and of group $\overline{A}$ onto group $\widetilde{A}$ and N is the full preimage by θ of subgroup $\widetilde{N}$, then N is a normal subgroup of finite index of group A and $f \notin (HgH)N$. Thus, the canonical homomorphism φ of group A onto quotient group A/N has the required property and the proposition is proven. □

4. Proof of Theorem 1.1

We prove first the following proposition.

Proposition 4.1. *If elements f and g of group G_{mn} such that $l(f) = l(g) > 1$ are not H-conjugate, then for some integer $t > 1$, elements $f\rho_t$ and $g\rho_t$ of group $G_{mn}(t)$ are not $H(t)$-conjugate.*

Proof. Let $f = x_1x_2\cdots x_r$ and $g = y_1y_2\cdots y_r$ be the reduced forms in group $G_{mn} = (A * B; H)$ of elements f and g.

We remind (see Proposition 2.3) that elements f and g are H-conjugate if and only if there exist elements $h_0, h_1, h_2, \ldots, h_r = h_0$ of H such that for any $i = 1, 2, \ldots, r$, we have

$$x_i = h_{i-1}^{-1} y_i h_i. \tag{4.1}$$

It then follows, in particular, that for each $i = 1, 2, \ldots, r$ elements x_i and y_i should lie in the same free factor A or B and define the same double coset modulo (H, H). So, we consider separately some cases.

Case 1. Suppose that for some index i elements x_i and y_i lie in different free factors A and B of group G_{mn} (and, certainly, are not in subgroup H). It follows from Proposition 3.4 that there exists an integer $t > 1$ such that elements $x_i\rho_t$ and $y_i\rho_t$ do not belong to the same free factor $A(t)$ or $B(t)$ of group $G_{mn}(t)$ (and, as above, lie in free factors of this group). Hence, by Proposition 2.3, in group $G_{mn}(t)$ elements $f\rho_t$ and $g\rho_t$ are not $H(t)$-conjugate.

Case 2. Let now for any $i = 1, 2, \ldots, r$ elements x_i and y_i belong to the same subgroup A or B and for some i element x_i does not belong to the double coset Hy_iH. By Proposition 3.6, there exists an integer $t_1 > 1$ such that for any positive integer t, divisible by t_1, element $x_i\rho_t$ is not in the double coset $H(t)(y_i\rho_t)H(t)$. From Proposition 3.5, there exist integers $t_2 > 1$ and $t_3 > 1$ such that for any positive integer t divisible by t_2 the length of element $f\rho_t$ in group $G_{mn}(t)$ is equal to r and for any positive integer t, divisible by t_3, the length of element $g\tau_t$ in group $G_{mn}(t)$ is equal to r. Thus, if $t = t_1t_2t_3$ then in group $G_{mn}(t)$, elements $f\rho_t$ and $g\rho_t$ have the reduced forms

$$(x_1\rho_t)(x_2\rho_t)\cdots(x_r\rho_t), \qquad (y_1\rho_t)(y_2\rho_t)\cdots(y_r\rho_t), \tag{4.2}$$

respectively and element $x_i\rho_t$ is not in the double coset $H(t)(y_i\rho_t)H(t)$. Again, by Proposition 2.3 these elements are not $H(t)$-conjugate.

Case 3. We now consider the case, when for any $i = 1, 2, \ldots, r$ elements x_i and y_i lie in the same free factor A or B and also determine the same double coset modulo (H, H). We prove some lemmas.

Lemma 4.2. *Let elements x and y belong to one of the groups A or B and do not belong to the subgroup H and $x \in HyH$, that is,*

$$x = c^{\alpha} d^{\beta} y c^{\gamma} d^{\delta} \tag{4.3}$$

for some integers α, β, γ and δ. If elements x and y belong to group A, then integers $\alpha + \gamma$, β and δ are uniquely determined by the equality (4.3). *If elements x and y belong to group B, then integers $\beta + \delta$, α and γ are uniquely determined by equality* (4.3).

Proof. Let elements x and y belong to subgroup A and let $x = c^{\alpha_1} d^{\beta_1} y c^{\gamma_1} d^{\delta_1}$ and $x = c^{\alpha_2} d^{\beta_2} y c^{\gamma_2} d^{\delta_2}$ for some integers α_1, β_1, γ_1, δ_1, α_2, β_2, γ_2 and δ_2. Rewriting the equality $c^{\alpha_1} d^{\beta_1} y c^{\gamma_1} d^{\delta_1} = c^{\alpha_2} d^{\beta_2} y c^{\gamma_2} d^{\delta_2}$ as $y^{-1} c^{\alpha_1-\alpha_2} d^{\beta_1-\beta_2} y = c^{\gamma_2-\gamma_1} d^{\delta_2-\delta_1}$, then, by Proposition 3.1, we conclude that $c^{\alpha_1-\alpha_2}d^{\beta_1-\beta_2} = c^{\gamma_2-\gamma_1}d^{\delta_2-\delta_1}$ and also that this element should belong to subgroup $\langle c\rangle$, that is, $\beta_1 - \beta_2 = \delta_2 - \delta_1 = 0$. So, we have $c^{\alpha_1-\alpha_2} = c^{\gamma_2-\gamma_1}$ and since the order of element c is infinite, then $\alpha_1 - \alpha_2 = \gamma_2 - \gamma_1$.

Thus, $\beta_1 = \beta_2$, $\delta_1 = \delta_2$ and $\alpha_1 + \gamma_1 = \alpha_2 + \gamma_2$ as it was required. The case when elements x and y belong to group B is esteemed similarly. □

Lemma 4.3. *Let elements x and y belong to one of groups $A(t)$ or $B(t)$ and do not belong to subgroup $H(t)$ and $x \in H(t)yH(t)$, that is,*

$$x = \left(cH^t\right)^{\alpha}\left(dH^t\right)^{\beta}y\left(cH^t\right)^{\gamma}\left(dH^t\right)^{\delta} \tag{4.4}$$

for some integers α, β, γ and δ. If elements x and y belong to group $A(t)$, then integers $\alpha + \gamma$, β and δ are uniquely determined modulo t by equality (4.4). *If elements x and y belong to group $B(t)$, then integers $\beta + \delta$, α and γ are uniquely determined modulo t by equality* (4.4).

The proof of Lemma 4.3 is completely similar to that of Lemma 4.2.

Lemma 4.4. *Let $f = x_1x_2\cdots x_r$ and $g = y_1y_2\cdots y_r$ be reduced elements of group G_{mn}, where $r > 1$ and let for every $i = 1, 2, \ldots, r$ the equality $x_i = u_iy_iv_i$ holds, for some elements u_i and v_i of subgroup H. Then there exists at most one sequence h_0, h_1, $h_2, \ldots, h_r$ of elements of subgroup H such that for any $i = 1, 2, \ldots, r$*

$$x_i = h_{i-1}^{-1}y_ih_i. \tag{4.5}$$

Moreover, if $u_i = c^{\alpha_i}d^{\beta_i}$ and $v_i = c^{\gamma_i}d^{\delta_i}$ for some integers α_i, β_i, γ_i and δ_i ($i = 1, 2, \ldots, r$) and $x_1, y_1 \in A$, then such sequence exists if, and only if, for any i, $1 < i < r$,

$$\begin{aligned} \alpha_i + \alpha_{i+1} + \gamma_{i-1} + \gamma_i = 0, &\quad \text{if } i \text{ is odd}, \\ \beta_i + \beta_{i+1} + \delta_{i-1} + \delta_i = 0, &\quad \text{if } i \text{ is even}. \end{aligned} \tag{4.6}$$

Proof. We suppose first that the sequence h_0, h_1, $h_2, \ldots, h_r$ of elements of subgroup H satisfying equality (4.5) exists, and let's write $h_i = c^{\mu_i}d^{\nu_i}$, for some integers μ_i and ν_i.

Then, since for any $i = 1, 2, \ldots, r$ the equality $h_{i-1}^{-1}y_ih_i = u_iy_iv_i$ holds, we have

$$y_i^{-1}(h_{i-1}u_i)y_i = h_iv_i^{-1}. \tag{4.7}$$

Since $r > 1$, then every element y_i, belonging to one of the subgroups A or B, does not lie in subgroup H, and consequently, from Proposition 3.1, for any $i = 1, 2, \ldots, r$, we have the equality $h_{i-1}u_i = h_iv_i^{-1}$. Substituting the expressions of elements h_i, u_i and v_i, we have for every $i = 1, 2, \ldots, r c^{\mu_{i-1}+\alpha_i}d^{\nu_{i-1}+\beta_i} = c^{\mu_i-\gamma_i}d^{\nu_i-\delta_i}$ and hence we obtain the system of numeric equations

$$\mu_{i-1} + \alpha_i = \mu_i - \gamma_i, \qquad \nu_{i-1} + \beta_i = \nu_i - \delta_i \quad (i = 1, 2, \ldots, r). \tag{4.8}$$

Since by (4.7) for every $i = 1, 2, \ldots, r$ element $h_iv_i^{-1}$ belongs to the intersection $y_i^{-1}Hy_i \cap H$ and, by supposition, elements y_1, $y_3, \ldots$ belong to subgroup A and elements y_2, $y_4, \ldots$ belong to subgroup B, then from Proposition 3.1, it follows that for odd i, we should have $h_{i-1}u_i = h_i v_i^{-1} \in \langle c\rangle$, and for even i, we should have $h_{i-1}u_i = h_i v_i^{-1} \in \langle d\rangle$. It means that for every odd i ($i = 1, 2, \ldots, r$) we have the equalities $\nu_{i-1} + \beta_i = 0$ and $\nu_i - \delta_i = 0$, and for every even i ($i = 1, 2, \ldots, r$) we have the equalities $\mu_{i-1} + \alpha_i = 0$ and $\mu_i - \gamma_i = 0$.

Hence, the values of the integers μ_i and ν_i are determined uniquely. Namely,

$$\mu_i = \begin{cases} \gamma_i & \text{if } i \text{ is even and } 2 \leqslant i \leqslant r, \\ -\alpha_{i+1} & \text{if } i \text{ is odd and } 1 \leqslant i \leqslant r-1, \end{cases} \tag{4.9}$$

$$\nu_i = \begin{cases} -\beta_{i+1} & \text{if } i \text{ is even and } 0 \leqslant i \leqslant r-1, \\ \delta_i & \text{if } i \text{ is odd and } 1 \leqslant i \leqslant r. \end{cases} \tag{4.10}$$

Moreover, from equalities (4.8) it follows, that

$$\begin{gathered} \mu_0 = -(\alpha_1 + \alpha_2 + \gamma_1), \\ \nu_r = \beta_r + \delta_{r-1} + \delta_r \quad \text{if } r \text{ is even}, \\ \mu_r = \alpha_r + \gamma_{r-1} + \gamma_r \quad \text{if } r \text{ is odd}. \end{gathered} \tag{4.11}$$

Thus, the statement that there can exist at most one sequence of elements h_0, h_1, $h_2, \ldots, h_r$ of subgroup H satisfying equalities (4.5) is demonstrated.

Substituting the value $\mu_{i-1} = \gamma_{i-1}$ and $\mu_i = -\alpha_{i+1}$ defined in (4.9) in the equalities $\mu_{i-1} + \alpha_i = \mu_i - \gamma_i$ of system (4.8), where $1 < i < r$ and i is odd, we obtain $\alpha_i + \alpha_{i+1} + \gamma_{i-1} + \gamma_i = 0$. Similarly, substituting the value $\nu_{i-1} = \delta_{i-1}$ and $\nu_i = -\beta_{i+1}$ defined in (4.10) in the equalities $\nu_{i-1} + \beta_i = \nu_i - \delta_i$ of systems (4.8), where $1 < i < r$ and i is even, we obtain $\beta_i + \beta_{i+1} + \delta_{i-1} + \delta_i = 0$.

Thus, under the existence in subgroup H of sequence h_0, h_1, $h_2, \ldots, h_r$ of elements satisfying (4.5), conditions (4.6) are satisfied.

Conversely, suppose conditions (4.6) are satisfied. Let

$$h_0 = c^{-(\alpha_1 + \alpha_2 + \gamma_1)} d^{-\beta_1} \tag{4.12}$$

and for all indexes i such that $1 \leqslant i < r$, we set

$$h_i = \begin{cases} c^{\gamma_i} d^{-\beta_{i+1}} & \text{if } i \text{ is even}, \\ c^{-\alpha_{i+1}} d^{\delta_i} & \text{if } i \text{ is odd}. \end{cases} \tag{4.13}$$

At last, for $i = r$ we set

$$h_r = \begin{cases} c^{\gamma_r} d^{\beta_r + \delta_{r-1} + \delta_r} & \text{if } r \text{ is even}, \\ c^{\alpha_r + \gamma_{r-1} + \gamma_r} d^{\delta_r} & \text{if } r \text{ is odd}. \end{cases} \tag{4.14}$$

Let us show that, the so-defined sequence of elements h_0, h_1, $h_2, \dots, h_r$ really fits to equalities (4.5).

If $i = 1$, using the permutability of elements c and y_1 we have

$$h_0^{-1} y_1 h_1 = c^{\alpha_1+\alpha_2+\gamma_1} d^{\beta_1} y_1 c^{-\alpha_2} d^{\delta_1} = c^{\alpha_1} d^{\beta_1} y_1 c^{\gamma_1} d^{\delta_1} = x_1. \tag{4.15}$$

If $1 < i < r$ and integer i is even, using the equality $\beta_i + \beta_{i+1} + \delta_{i-1} + \delta_i = 0$ and permutability of elements d and y_i, we have

$$h_{i-1}^{-1} y_i h_i = c^{\alpha_i} d^{-\delta_{i-1}} y_i c^{\gamma_i} d^{-\beta_{i+1}} = c^{\alpha_i} y_i c^{\gamma_i} d^{-(\delta_{i-1}+\beta_{i+1})} = c^{\alpha_i} y_i c^{\gamma_i} d^{\beta_i+\delta_i} = c^{\alpha_i} d^{\beta_i} y_i c^{\gamma_i} d^{\delta_i} = x_i. \tag{4.16}$$

If $1 < i < r$ and integer i is odd, using the equality $\alpha_i+\alpha_{i+1}+\gamma_{i-1}+\gamma_i = 0$ and permutability of elements c and y_i, we have

$$h_{i-1}^{-1} y_i h_i = c^{-\gamma_{i-1}} d^{\beta_i} y_i c^{-\alpha_{i+1}} d^{\delta_i} = c^{-(\gamma_{i-1}+\alpha_{i+1})} d^{\beta_i} y_i d^{\delta_i} = c^{\alpha_i+\gamma_i} d^{\beta_i} y_i d^{\delta_i} = c^{\alpha_i} d^{\beta_i} y_i c^{\gamma_i} d^{\delta_i} = x_i. \tag{4.17}$$

If integer r is even, then

$$h_{r-1}^{-1} y_r h_r = c^{\alpha_r} d^{-\delta_{r-1}} y_r c^{\gamma_r} d^{\beta_r+\delta_{r-1}+\delta_r} = c^{\alpha_r} d^{\beta_r} y_r c^{\gamma_r} d^{\delta_r} = x_r, \tag{4.18}$$

and if r is odd, then

$$h_{r-1}^{-1} y_r h_r = c^{-\gamma_{r-1}} d^{\beta_r} y_r c^{\alpha_r+\gamma_{r-1}+\gamma_r} d^{\delta_r} = c^{\alpha_r} d^{\beta_r} y_r c^{\gamma_r} d^{\delta_r} = x_r. \tag{4.19}$$

So, Lemma 4.4 is completely demonstrated. □

Similar argument gives the following lemma.

Lemma 4.5. *Let $f = x_1 x_2 \cdots x_r$ and $g = y_1 y_2 \cdots y_r$ be reduced elements of group $G_{mn}(t)$, where $r > 1$, and let for every $i = 1, 2, \dots, r$ the equality $x_i = u_i y_i v_i$ takes place, for some elements u_i and v_i of subgroup $H(t)$. Then there exists at most one sequence h_0, h_1, $h_2, \dots, h_r$ of elements of subgroup $H(t)$ such that for any $i = 1, 2, \dots, r$, one has*

$$x_i = h_{i-1}^{-1} y_i h_i. \tag{4.20}$$

Moreover, if $u_i = (cH^t)^{\alpha_i} (dH^t)^{\beta_i}$ and $v_i = (cH^t)^{\gamma_i} (dH^t)^{\delta_i}$ for some integers α_i, β_i, γ_i and δ_i ($i = 1, 2, \dots, r$) and $x_1, y_1 \in A(t)$, then such sequence exists if and only if for any integer i, $1 < i < r$, one has

$$\begin{aligned} &\alpha_i + \alpha_{i+1} + \gamma_{i-1} + \gamma_i \equiv 0 \pmod{t} \quad \text{if } i \text{ is odd}, \\ &\beta_i + \beta_{i+1} + \delta_{i-1} + \delta_i \equiv 0 \pmod{t} \quad \text{if } i \text{ is even}. \end{aligned} \tag{4.21}$$

We can now end the consideration of Case 3 and thus complete the proof of Proposition 4.1.

By Proposition 2.3, elements f and g are H-conjugate if and only if their cyclic permutation $x_2 \cdots x_r x_1$ and $y_2 \cdots y_r y_1$ are H-conjugate. Therefore, we can suppose, without loss of generality, that elements x_1 and y_1 belong to subgroup A.

By supposition, for every $i = 1, 2, \ldots, r$ there exist integers α_i, β_i, γ_i and δ_i such that

$$x_i = c^{\alpha_i} d^{\beta_i} y_i c^{\gamma_i} d^{\delta_i}. \tag{4.22}$$

If integer i, $1 < i < r$, is even, then $y_i \in B$, $y_{i-1}, y_{i+1} \in A$ and consequently, from Lemma 4.2, integers $\beta_i + \delta_i$, δ_{i-1} and β_{i+1} do not depend from the particular expressions of the elements x_i (in the form $x_i = u y_i v$, where $u, v \in H$), and these integers are uniquely determined by the sequences $x_1, x_2, \ldots, x_r$ and $y_1, y_2, \ldots, y_r$. Similarly, for any odd i, $1 < i < r$, integers $\alpha_i + \gamma_i$, γ_{i-1} and α_{i+1} are uniquely determined by these sequences. It means, in turn, that the satisfiability of conditions (4.6) of Lemma 4.4 depends only on the sequences x_1, $x_2, \ldots, x_r$ and $y_1, y_2, \ldots, y_r$.

Let's now conditions (4.6) of Lemma 4.4 are not satisfied, that is, either for some even i, $1 < i < r$, the sum $\beta_i + \beta_{i+1} + \delta_{i-1} + \delta_i$ is different from zero, or for some odd i, $1 < i < r$, the sum $\alpha_i + \alpha_{i+1} + \gamma_{i-1} + \gamma_i$ is different from zero. Then it is possible to find an integer $t_1 > 1$, not dividing the respective sum. Let's also choose, according to Proposition 3.5, the integers $t_2 > 1$ and $t_3 > 1$ such that for all positive integer t, divisible by t_2, the length of element $f\rho_t$ in group $G_{mn}(t)$ is equal to r and for all positive integer t, divisible by t_3, the length of element $g\rho_t$ in group $G_{mn}(t)$ is equal to r. Then if $t = t_1 t_2 t_3$, in group $G_{mn}(t)$, elements $f\rho_t$ and $g\rho_t$ have reduced forms

$$(x_1\rho_t)(x_2\rho_t)\cdots(x_r\rho_t), \qquad (y_1\rho_t)(y_2\rho_t)\cdots(y_r\rho_t), \tag{4.23}$$

respectively. Further, for every $i = 1, 2, \ldots, r$,

$$x_i\rho_t = (cH^t)^{\alpha_i}(dH^t)^{\beta_i}(y_i\rho_t)(cH^t)^{\gamma_i}(dH^t)^{\delta_i}. \tag{4.24}$$

From Lemma 4.3 and the selection of integer t_1 it follows that for elements $f\rho_t$ and $g\rho_t$, conditions (4.21) of Lemma 4.5 are not satisfied. Therefore from this lemma and Proposition 2.3, elements $f\rho_t$ and $g\rho_t$ are not $H(t)$-conjugate in group $G_{mn}(t)$.

Let now the reduced forms of elements f and g satisfy conditions (4.6) of Lemma 4.4. Then according to this lemma, in subgroup H there exists the only sequence of elements h_0, $h_1, h_2, \ldots, h_r$ such that for any $i = 1, 2, \ldots, r$, we have $x_i = h_{i-1}^{-1} y_i h_i$. Since elements f and g are not H-conjugate, then elements h_0 and h_r should be different. Since group G_{mn} is residually finite, by Proposition 3.3 there exists an integer $t_1 > 1$ such that $h_0\rho_{t_1} \neq h_r\rho_{t_1}$. Let's choose one more integer t_2 such that in group $G_{mn}(t_2)$ elements $f\rho_{t_2}$ and $g\rho_{t_2}$ have length r. Then if $t = t_1 t_2$, in group $G_{mn}(t)$, $h_0\rho_t \neq h_r\rho_t$, elements $f\rho_t$ and $g\rho_t$ have reduced forms

$$(x_1\rho_t)(x_2\rho_t)\cdots(x_r\rho_t), \qquad (y_1\rho_t)(y_2\rho_t)\cdots(y_r\rho_t), \tag{4.25}$$

respectively and for every $i = 1, 2, \ldots, r$

$$x_i\rho_t = (cH^t)^{\alpha_i}(dH^t)^{\beta_i}(y_i\rho_t)(cH^t)^{\gamma_i}(dH^t)^{\delta_i}. \tag{4.26}$$

Moreover, in group $G_{mn}(t)$, for any $i = 1, 2, \ldots, r$, the equality

$$x_i\rho_t = \left(h_{i-1}\rho_t\right)^{-1}\left(y_i\rho_t\right)\left(h_i\rho_t\right) \tag{4.27}$$

holds.

Since the sequence $h_0\rho_t, h_1\rho_t, h_2\rho_t, \ldots, h_r\rho_t$ is, by Lemma 4.5, the unique sequence of elements of subgroup $H(t)$ satisfying these equalities, then from Proposition 2.3, elements $f\tau_t$ and $g\tau_t$ are not $H(t)$-conjugate in group $G_{mn}(t)$.

Hence Proposition 4.1 is completely demonstrated. □

We now proceed directly to the proof of Theorem 1.1.

As remarked above, for any $t > 1$, group $G_{mn}(t)$ is conjugacy separable. Therefore, for the proof of the Theorem it is enough to show that for any two nonconjugate in group G_{mn} elements f and g of group G_{mn}, there exists an integer $t > 1$ such that elements $f\rho_t$ and $g\rho_t$ are not conjugate in group $G_{mn}(t)$. To this end we make use of the conjugacy criterion of elements of groups G_{mn} and $G_{mn}(t)$ given in Proposition 3.2.

So, let $f = x_1x_2\cdots x_r$ and $g = y_1y_2\cdots y_r$ be the reduced forms in group $G_{mn} = (A * B;\ \ H)$ of two nonconjugate in group G_{mn} elements f and g. By Proposition 3.2 we can assume that elements f and g are cyclically reduced. In accordance with this proposition we must consider separately some cases.

Case 1 ($l(f) \neq l(g)$). From Proposition 3.5, there exists an integer $t_1 > 1$ such that for any positive integer t, divisible by t_1, the length of element $f\rho_t$ in group $G_{mn}(t)$ coincides with the length of element f in group G_{mn}. Similarly, there exists integer $t_2 > 1$ such that for any positive integer t, divisible by t_2, the length of element $g\rho_t$ in group $G_{mn}(t)$ coincides with the length of element g in group G_{mn}. Thus, if $t = t_1t_2$ elements $f\rho_t$ and $g\rho_t$ of group $G_{mn}(t)$ are, as it is easy to see, cyclically reduced and have different length. Hence, by Proposition 3.2, these elements are not conjugate in this group. Thus integer t is the required.

Case 2 ($l(f) = l(g) = 1$). In this case each of these elements belongs to one of the subgroups A or B. Suppose first that both elements lie in the same of these subgroups; let it be, for instance, subgroup A. Since elements f and g are not conjugate in group A and group A is conjugacy separable [3], then by Proposition 3.3 there exists an integer $t > 1$ such that in group $A(t)$ elements $f\rho_t$ and $g\rho_t$ are not conjugate. Proposition 3.2 now implies that elements $f\rho_t$ and $g\rho_t$ are not conjugate in group $G_{mn}(t)$.

Let now element f belongs to subgroup A, element g belongs to subgroup B and these elements do not belong to subgroup H. By Proposition 3.4, there exist integers $t_1 > 1$ and $t_2 > 1$ such that for any positive integer t, divisible by t_1, element $f\rho_t$ does not belong to subgroup $H\rho_t$, and for any positive integer t, divisible by t_2, element $g\rho_t$ does not belong to subgroup $H\rho_t$. Then if $t = t_1t_2$, Proposition 3.2 implies that elements $f\rho_t$ and $g\rho_t$ are not conjugate in group $G_{mn}(t)$.

Case 3 ($l(f) = l(g) > 1$). Let $g_i = y_iy_{i+1}\cdots y_ry_1\cdots y_{i-1}$ ($i = 1, 2, \ldots, r$) be all the cyclic permutations of element g. Since elements g and g_i are conjugate and elements f and g are not conjugate, element f is not H-conjugate to any of elements $g_1, g_2, \ldots, g_r$. It follows from Proposition 4.1 that for every $i = 1, 2, \ldots, r$, there exists an integer $t_i > 1$ such that elements $f\rho_{t_i}$ and $g_i\rho_{t_i}$ are not $H(t_i)$-conjugate in group $G_{mn}(t_i)$. Let also integer $t_0 > 1$ be chosen such that for all positive integer t, divisible by t_0, elements $f\rho_t$ and $g\rho_t$ have length r in group $G_{mn}(t)$. Then if $t = t_0 t_1 \cdots t_r$ in group $G_{mn}(t)$, elements $f\rho_t$ and $g_i\rho_t$ have reduced forms

$$\begin{gathered}(x_1\rho_t)(x_2\rho_t)\cdots(x_r\rho_t)\\(y_i\rho_t)(y_{i+1}\rho_t)\cdots(y_r\rho_t)(y_1\rho_t)\cdots(y_{i-1}\rho_t),\end{gathered} \tag{4.28}$$

respectively. Furthermore, elements $f\rho_t$ and $g_i\rho_t$ are not $H(t)$-conjugate in group $G_{mn}(t)$. Since an arbitrary cyclic permutation of element $g\rho_t$ coincides with some element $g_i\rho_t$, then from Proposition 3.2 it follows that elements $f\rho_t$ and $g\rho_t$ are not conjugate in group $G_{mn}(t)$.

The proof of Theorem is now complete.

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